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ANOTHER CHARACTERIZATION OF THE CENTROID OF A TREE

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A particular notion of centrality in trees, the centroid, is studied. Previous work has shown that the centroid and the distance center are equivalent. A new notion of centrality called the telephone center is introduced. This is shown to be equivalent to the centroid as well.

1. Introduction

The notion of the "center" of a tree has been much studied in the literature. A representative sample of results on this topic can be found in papers [1–3, 6, 7]. A related, but distinct, notion of centrality is the "centroid" of a tree, which was first defined by Jordan [4] in 1879.

Recent papers by Zelinka [8] and Kang and Ault [5] and Goldman [1] have shown that the centroid is equivalent to what is called the distance center of a tree. Both Kang and Ault and Goldman have also given linear-time algorithms for finding the centroid of a tree.

In this note we introduce a new notion of centrality called the telephone center of a graph. We show that the telephone center of a tree T is also equivalent to the centroid of T.

2. The centroid of a tree

A *tree* is a connected acyclic graph. An *endvertex* u in a tree T is a vertex of degree one. A *branch* of a vertex v of a tree T is a maximal subtree of T containing v as an endvertex. The *weight* $\mathrm{wt}(v)$ of a vertex v of T is the maximum number of edges of any branch at v. A vertex u is a *centroid vertex* of T if u has minimum weight; the *centroid* of T is composed of all centroid vertices.

Zelinka [8], Kang and Ault [5] and Goldman have independently obtained a characterization of the centroid of a tree T based on distances between vertices.

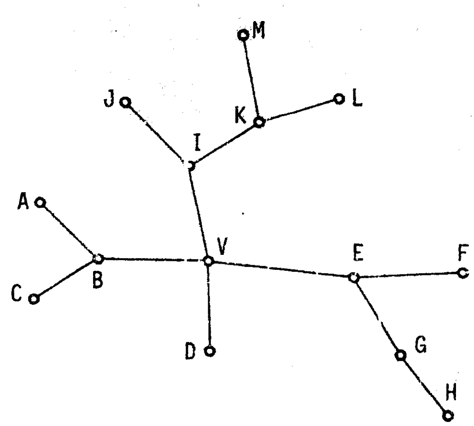
Define $d(u, v)$ to be the length of the shortest path between vertices u and v in T. Then the *distance of vertex* u, denoted $L(u)$, is the sum of all $d(u, v)$ in T, i.e.

$L(u)=\sum_{v\in T} d(u, v)$. The *distance center* of a tree T consists of those vertices u for which $L(u)$ is minimum. The distance center of T is identical to the centroid of T.

3. The telephone center of a tree

Let the vertices of a tree T represent telephone lines; and let a path represent a telephone call between its endvertices. Assume that at any given time, a vertex can be involved in only one call. Define the *switchboard number of vertex* v, denoted sb(v), to be the maximum number of calls which can pass through v at any given time. The *telephone center* of a tree T consists of those vertices v of T for which sb(v) is maximum.

Fig. 1 illustrates the switchboard numbers for all the vertices in a tree T and a set of calls which can pass through the vertex v. Note that v is the only vertex in the telephone center of T.



Calls through V

EJ
FI
GK
HL
AM
BD

Fig. 1.

The next results show that a vertex having the maximum switchboard number in a tree is also a vertex having minimum distance, i.e. the telephone center and the centroid of a tree are identical.

Lemma 1. *For every vertex v in T,* $\mathrm{sb}(v) \leqslant (n-1)/2$.

Theorem 2. *Let u be an arbitrary vertex in a tree T having n vertices; let the vertices adjacent to u be $u_1, u_2, \ldots, u_k$; let the branches of u containing $u_1, u_2, \ldots, u_k$ be $B_1, B_2, \ldots, B_k$, respectively; let $n_1, n_2, \ldots, n_k$ be the number of vertices in $B_1, B_2, \ldots, B_k$, respectively, where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$; let $B' = \bigcup_{i=1}^{k-1} B_i$; and let $n' = \sum_{i=1}^{k-1} n_i$.*

Then (i) *if $n_k \geqslant n'$, then* $\mathrm{sb}(u) = n' = n - 1 - n_k$. *and* (ii) *if $n_k < n'$, then* $\mathrm{sb}(u) = \lfloor \frac{1}{2}(n-1) \rfloor$.

Proof. (i) Clearly every call (path) through vertex u must contain a vertex in B', thus $\mathrm{sb}(u) \leqslant |B'| = n'$. But if $n_k \geqslant n'$, then every vertex in B' can be paired with a vertex in B_k, i.e. n' calls can pass through u and $\mathrm{sb}(u) = n'$.

(ii) Conversely if $n_k < n'$ then all vertices, except possibly one, can be paired up with each other to produce $\lfloor \frac{1}{2}(n-1) \rfloor$ calls through vertex u. This can be done by repeatedly pairing up two vertices in different branches, until either (a) all vertices (except possibly one) are paired off, or (b) a point is reached where all vertices in all branches except one, say B_k, have been paired off; and there remain several vertices in B_k which are not paired off.

For each pair of such vertices in B_k, say u_k and v_k, we can find two vertices which have been paired off, say (u_i, v_j) where neither u_i nor v_j are in B_k. We can then produce two new calls (u_k, u_i) and (v_k, v_j) and reduce the number of unpaired vertices in B_k by two. We can continue in this fashion until either all vertices in B_k (except possibly one) are paired off, giving us $\lfloor \frac{1}{2}(n-1) \rfloor$ calls through u, or there still remain unpaired vertices in B_k and every call is between a vertex in B_k. But this implies that $n_k \geqslant n'$, contradicting our assumption that $n_k < n'$.

Corollary 3. *A vertex u is in the telephone center of a tree T if and only if it is in the centroid of T.*

Proof. Clearly by Theorem 9 $\mathrm{sb}(u) = \lfloor \frac{1}{2}(n-1) \rfloor$ if and only if $n_k < n'$. But $n_k < n'$ if and only if every branch of u satisfies $n_i \leqslant n/2$; and by a result in [5] this means that vertex u is in the centroid. Thus $\mathrm{sb}(n) = \lfloor \frac{1}{2}(n-1) \rfloor$ if and only if u is a centroid vertex.

Finally by Lemma 1 we can conclude that any vertex u for which $\mathrm{sb}(u) = \lfloor \frac{1}{2}(n-1) \rfloor$ must be in the telephone center, and conversely.

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